Set Up

- $g(t)$ solves KRF for $t \in [0,T)$, $T < \infty$
- $\pi : X \to Y$ blows down $E \mapsto y_0$
- $[\omega_0] + T c_1(K_X) = [\pi^* \omega_Y]$ (Cohomological obstruction)

Reference Metrics

line from ω_0 to $\pi^*\omega_Y$ at speed T

given sol. KRF

- $\hat{\omega}_t = \frac{1}{T}\{(T-t)\omega_0 + t\pi^*\omega_Y\} \in [\omega(t)] = [\omega_0] + t c_1(K_X)$
- $\frac{d}{dt}(\hat{\omega}_t) \in c_1(K_X)$ $\left(\text{Note: } \frac{d}{dt}(\hat{\omega}_t) = \underbrace{\frac{1}{T}(\pi^*\omega_Y - \omega_0)}_{\text{a fixed form}}\right)$

$\Rightarrow \exists!\ \Omega$ volume form :

- $\frac{i}{2\pi}\partial\bar{\partial}\log\Omega = \frac{\partial}{\partial t}(\hat{\omega}_t) = \eta$ $\quad \eta$
- $\int_X \Omega = 1$

Equation

- Our initial equation is : $\boxed{\omega'(t) = -Ric(\omega)}$
- In our case $\omega(t) = \hat{\omega}_t + \frac{i}{2\pi}\partial\bar{\partial}\varphi(t)$ for some $\varphi(t)$

- Thus our equation becomes

$$\left(\hat{\omega}_t + \frac{i}{2\pi}\partial\bar{\partial}\varphi(t)\right)' = -\operatorname{Ric}\left(\hat{\omega}_t + \frac{i}{2\pi}\partial\bar{\partial}\varphi(t)\right)$$

$$\Rightarrow \frac{i}{2\pi}\partial\bar{\partial}\log\Omega + \frac{i}{2\pi}\partial\bar{\partial}\varphi'(t) = \frac{i}{2\pi}\log\left(\hat{\omega}_t + \frac{i}{2\pi}\partial\bar{\partial}\varphi(t)\right)^n$$

~~(Here choose~~

- Using $\partial\bar{\partial}$-Lemma get:

$$\boxed{\varphi'(t) = \log\frac{\left(\hat{\omega}_t + \frac{i}{2\pi}\partial\bar{\partial}\varphi(t)\right)^n}{\Omega}}$$

- Fix the constant of φ by:

$$\boxed{\varphi|_{t=0} \equiv 0}$$

Thus the goal becomes to get estimates for φ!

Lemma 2.1 :

(i) $\|\varphi\|_{L^\infty} \leq C$

(ii) $\dot{\varphi} \leq C$

(iii) $\omega^n \leq C\Omega$

positive, closed

$\omega_t \to \omega(T)$ weakly as currents

(iv) $\varphi(t) \to \varphi(T)$ (pointwise converg.) w/ $\omega_T := \hat{\omega}_T + \frac{i}{2\pi}\partial\bar{\partial}\varphi(T) \geq 0$

Proof:

(i) • First exists C_0 uniform s.t. $\hat{\omega}_t \leq C_0\Omega$

• $\psi := \varphi - (\log C_0 + 1)t$

Assume $\sup\limits_{X\times[0,t_0]} \psi$ is taken at $(x_0, t_0) \in X \times (0,T)$.

Then at (x_0, t_0) have $i\partial\bar{\partial}\varphi \leq 0$ and:

$$0 \leq \frac{\partial\psi}{\partial t} = \log \frac{(\hat{\omega}_t + \frac{i}{2\pi}\partial\bar{\partial}\varphi)^n}{\Omega} - \log C_0 - 1$$

$$\leq \log \frac{(\hat{\omega}_t)^n}{\Omega} - \log C_0 - 1$$

$$\leq -1$$

Contradiction gives $\Rightarrow$ $\boxed{\varphi \leq C}$

(Specifically, $\varphi \leq (\log C_0)t$, $t \leq T$)

Point: $\cdot$ $\varphi(0) \equiv 0$

$\cdot$ ~~max~~ At max $\varphi(t)$,

neg!

$$\frac{\partial \varphi}{\partial t} = \log \frac{(\hat{\omega}_t + i\partial\bar{\partial}\varphi(t))^n}{\Omega}$$

$$\leq \log \frac{\hat{\omega}_t}{\Omega}$$

$$\leq \log C_0.$$

I.e. $\max\varphi(t)$ grows slower than $\log C_0$!

<u>Lower bound:</u>

Now at $\min \varphi(t)$ have $\frac{i}{2\pi}\partial\bar{\partial}\varphi(t) \geq 0$. Therefore:

$$\frac{\partial \varphi}{\partial t} = \log \frac{(\omega_t + i\partial\bar{\partial}\varphi(t))^n}{\Omega}$$

$$\geq \log \frac{\omega_t^n}{\Omega}$$

Since $\hat{\omega}_t^n = \frac{1}{T}\left((T-t)\omega_0 + t\omega_Y\right)^n$

$$\geq \left(\frac{T-t}{T}\right)^n \omega_0^n$$

Get then that

$$\frac{\partial \varphi}{\partial t} \geq \log \frac{\left(\frac{T-t}{T}\right)^n \omega_0^n}{\Omega} \quad \text{at min}$$

$$\geq \log\left[\frac{\omega_0^n}{\Omega}\left(\frac{T-t}{T}\right)^n\right] \qquad \frac{\omega_0^n}{\Omega}\frac{1}{T} \geq C_0$$

$$\geq \log C_0 + n \log (T-t).$$

So $\dfrac{\partial \varphi}{\partial t} \geq \log C_0 + n \log (T-t)$ at min point.

So if you integrate you get

$$\min \varphi(t) \geq \int_0^t \left[\log C_0 + n \log (T-t)\right] dt$$

$$= (\log C_0) t + n \int_0^t \log (T-t)\, dt$$

$$= (\log C_0) t - n\left[(T-t)\log(T-t) - (T-t)\right]\Big|_0^t$$

But the last part does not blow up as $t \to T$!

So $\min \varphi(t)$ is bounded below!

(ii) So now have shown that

- $\dot{\varphi} \leq \log C_0$, at max point of $\varphi(t)$
- $\dot{\varphi} \geq \log c_0 + n \log (T-t)$, at min of $\varphi(t)$

So to try to get something like

$$\underbrace{-C \leq}_{\text{prop fucked}} \underbrace{\dot{\varphi} \leq C}_{\text{maybe okay}}, \text{ everywhere}$$

So for the second estimate you might like some equation like (heat equation)$(\dot{\varphi}) \leq 0$. Then the max decreases in time.

- $\hat{\omega}_t = \omega_0 + t\underbrace{\left\{\frac{1}{T}(\pi^*\omega_r - \omega_0)\right\}}_{\eta}, \quad 0 \leq t < T$
- $\omega = \hat{\omega}_t + \frac{i}{2\pi}\partial\bar{\partial}\varphi(t)$

Compute:

- $\frac{\partial}{\partial t}(\dot{\varphi}) = \frac{\partial}{\partial t}\left(\log \frac{\omega^n}{\Omega}\right)$ ind of time

$$= \frac{\dot{\omega} \wedge n\omega^{n-1}}{\omega^n} \quad \text{trace}$$

$$= \frac{\eta \wedge n\omega^{n-1}}{\omega^n} + \underbrace{\frac{\frac{i}{2\pi}\partial\bar{\partial}\dot{\varphi}(t) \wedge n\omega^{n-1}}{\omega^n}}_{\Delta_\omega(\dot{\varphi})}$$

So then have:

- $$\left(\frac{\partial}{\partial t} - \Delta_\omega\right)(\dot{\varphi}) = tr_\omega(\eta)$$

But we don't know that $\eta = \frac{1}{T}(\pi^* \omega_Y - \omega_0)$ necessarily must be negative! So try adding in a t!

- $$\left(\frac{\partial}{\partial t} - \Delta_\omega\right)(t\dot{\varphi}) = t\, tr_\omega(\eta) + \dot{\varphi}$$

Try to get rid of $\dot{\varphi}$!

- $$\left(\frac{\partial}{\partial t} - \Delta_\omega\right)(t\dot{\varphi} - \varphi) = t\, tr_\omega(\eta) + \dot{\varphi} - \dot{\varphi} + \Delta_\omega(\varphi)$$
$$= tr_\omega\left(t\eta + \frac{i}{2\pi}\partial\bar{\partial}\varphi\right)$$

But last part is good! Remember

- $$\omega(t) = \omega_0 + t\eta + \frac{i}{2\pi}\partial\bar{\partial}\varphi(t)$$

$$\Rightarrow \underbrace{tr_\omega\, \omega}_{n} = \underbrace{tr_\omega\, \omega_0}_{\text{positive}} + tr_\omega\left(t\eta + \frac{i}{2\pi}\partial\bar{\partial}\varphi\right)$$

So have then that

$$\left(\frac{\partial}{\partial t} - \Delta_\omega\right)(t\dot{\varphi} - \varphi) = \underbrace{n - tr_\omega \omega_0}_{\text{const over manifold}}$$

Therefore have

$$\left(\frac{\partial}{\partial t} - \Delta\right)(t\dot{\varphi} - \varphi - nt) = -tr_\omega \omega_0 \leq 0$$

So $t\dot{\varphi} - \varphi - nt \leq \max -\varphi(0)$ which is bounded!

Thus have $\dot{\varphi} \leq \frac{C}{t}$. However $\varphi(0) \equiv 0$ so $\varphi(0)$ is max at all points so $\dot{\varphi}(0) \leq \log c_0$ for all points on X. Thus $\dot{\varphi} \leq C$ for all t!

Note: We don't have an effective bound from this argument (which is not needed).

Now for (iii):

$$\bullet\ (\omega^n)' = \mathring{\omega} \wedge n\omega^{n-1}$$

$$= \frac{(\mathring{\omega} \wedge n\omega^{n-1})\,\omega^n}{\omega^n}$$

$\omega = \omega_0 + t\eta + \frac{i}{2\pi}\partial\bar{\partial}\varphi(t)$

$$= \mathrm{tr}_\omega(\mathring{\omega})\,\omega^n$$

$$= \mathrm{tr}_\omega\left(\eta + \frac{i}{2\pi}\partial\bar{\partial}\mathring{\varphi}(t)\right)\omega^n$$

$$= \left(\mathrm{tr}_\omega(\eta) + \Delta_\omega(\mathring{\varphi})\right)\omega^n$$

$$= \frac{\partial\mathring{\varphi}}{\partial t}\,\omega^n$$

So using the bound from (ii) you get that

$$(\omega^n)' \leq C\omega^n, \quad \forall t \in (0,T)$$

Thus integrating gives you: fixed and unif equiv!

$$\omega^n \leq C'\omega_0^n \leq C\Omega$$

(iv) Now since $\|\varphi\|_{L^\infty} \leq C$, we get that $\varphi(t) \to \varphi(T)$ along a subsequence (possible that limit not unique?).

[* Question: Know $\dot{\varphi} \leq C$ but don't know $\dot{\varphi} \geq -C$. So how to say limit unique? Does Arzela-Ascoli somehow apply? Is convergence uniform? Must limiting function be continuous? *]

Once those questions are answered, (iv) just follows from standard (ha!) current theory! $\boxtimes$

Lemma 2.2: The solution $\omega(t)$ of KRF satisfies:

(i) $\exists c > 0$ uniform so that $\omega \geq c\pi^*\omega_Y$

(ii) For $K \subset X \setminus E$, $\exists C_{K,i}$ s.t. $\|\omega\|_{\mathcal{C}^i(K)} \leq C_{K,i}$

(iii) ω_T in Lemma 2.1 is smooth Kähler on $X \setminus E$

(iv) $\omega(t) \to \omega_T$ in $\mathcal{C}^\infty$ on compact sets in $X \setminus E$

Proof: (i) First let $u = \mathrm{tr}_g(\pi^* g_Y)$. Then if $\omega_Y = h_{\bar\beta\alpha}\, dz^\alpha \wedge d\bar z^\beta$, we have

$$\Delta u = g^{k\bar l}\partial_k \partial_{\bar l}\left(g^{i\bar j}\, \pi^\alpha_i\, \pi^{\bar\beta}_{\bar j}\, h_{\bar\beta\alpha}\right)$$

$$= g^{i\bar l} g^{k\bar j} R_{\bar l k}\, \pi^\alpha_i\, \pi^{\bar\beta}_{\bar j}\, h_{\bar\beta\alpha} + g^{i\bar j} g^{k\bar l}\, \pi^\alpha_{i,k}\, \pi^{\bar\beta}_{\bar j,\bar l}\, h_{\bar\beta\alpha}$$

$$- g^{i\bar j} g^{k\bar l} R(h)_{\bar\delta\gamma\bar\beta\alpha}\, \pi^\alpha_i\, \pi^{\bar\beta}_{\bar j}\, \pi^\gamma_k\, \pi^{\bar\delta}_{\bar l}$$

(See attached computation.) $R_{\bar l k}$ is $Ric(g)$ and $R(h)_{\bar\delta\gamma\bar\beta\alpha}$ is Riemannian (h).

However, h is fixed! So

$$R(h)_{\bar\delta\gamma\bar\beta\alpha}\, \pi^\alpha_i\, \pi^{\bar\beta}_{\bar j}\, \pi^\gamma_k\, \pi^{\bar\delta}_{\bar l} \le C\, h_{\bar\beta\alpha}\, h_{\bar\delta\gamma}\, \pi^\alpha_i\, \pi^\gamma_k\, \pi^{\bar\beta}_{\bar j}\, \pi^{\bar\delta}_{\bar l},$$

where C depends only on h. Thus we have:

$$\Delta u \ge g^{i\bar l} g^{k\bar j} R_{\bar l k}\, \pi^\alpha_i\, \pi^{\bar\beta}_{\bar j}\, h_{\bar\beta\alpha} + g^{i\bar j} g^{k\bar l}\, \pi^\alpha_{i,k}\, \pi^{\bar\beta}_{\bar j,\bar l}\, h_{\bar\beta\alpha} - C u^2$$

Now the KRF equation gives us:

$$\frac{\partial}{\partial t}(u) = \frac{\partial}{\partial t}\left(g^{i\bar{j}}\pi^{\alpha}_{i}\pi^{\bar{\beta}}_{\bar{j}}h_{\bar{\beta}\alpha}\right)$$

$$= -g^{i\bar{l}}\dot{g}_{\bar{l}k}g^{k\bar{j}}\pi^{\alpha}_{i}\pi^{\bar{\beta}}_{\bar{j}}h_{\bar{\beta}\alpha}$$

$$= g^{i\bar{l}}R_{\bar{l}k}g^{k\bar{j}}\pi^{\alpha}_{i}\pi^{\bar{\beta}}_{\bar{j}}h_{\bar{\beta}\alpha}$$

Thus we have:

$$\left(\frac{\partial}{\partial t} - \Delta\right)(u) \leq -g^{i\bar{j}}g^{k\bar{l}}\pi^{\alpha}_{i,k}\pi^{\bar{\beta}}_{\bar{j},\bar{l}}h_{\bar{\beta}\alpha} + Cu^2$$

Now note that:

$$\left(\frac{\partial}{\partial t} - \Delta\right)(\log u) = \frac{\dot{u}}{u} - \frac{\Delta(u)}{u} + \frac{|\nabla(u)|^2}{u^2}$$

$$= \frac{1}{u}\left(\frac{\partial}{\partial t} - \Delta\right)(u) + \frac{|\nabla(u)|^2}{u^2}$$

$$\leq \frac{1}{u}\left[-g^{i\bar{j}}g^{k\bar{l}}\pi^{\alpha}_{i,k}\pi^{\bar{\beta}}_{\bar{j},\bar{l}}h_{\bar{\beta}\alpha} + \frac{|\nabla(u)|^2}{u}\right] + Cu$$

$$\overset{(*)}{\leq} Cu$$

The last step follows because this (claim!) is negative!

So now have an equation like this:

$$\left(\frac{\partial}{\partial t} - \Delta\right)(\log u) \leq Cu \quad \longleftarrow \text{need to kill that!}$$

Since $u = \operatorname{tr}_\omega(\pi^*\omega_Y)$, it suffices if we kill a larger form. We've seen the term $\operatorname{tr}_\omega(\hat{\omega}_t)$ before so try that. Note $\operatorname{tr}_\omega(\hat{\omega}_t) \geq C' \operatorname{tr}_\omega(\pi^*\omega_Y)$.

What gave us that term?

- $\operatorname{tr}_\omega(\omega) = n = \operatorname{tr}_\omega(\hat{\omega}_t) + \Delta(\varphi)$

So use this term! Compute:

$$\left(\frac{\partial}{\partial t} - \Delta\right)(\log u - A\varphi) \leq Cu - A\dot{\varphi} + A\Delta(\varphi)$$

$$= Cu - A\log\frac{\omega^n}{\Omega} + An - A\operatorname{tr}_\omega(\hat{\omega}_t)$$

So first assume A is big enough (dep on C) to kill this term. Then have

$$\left(\frac{\partial}{\partial t} - \Delta\right)(\log u - A\varphi) \leq A\log\frac{\Omega}{\omega^n} + An - A\operatorname{tr}_\omega(\hat{\omega}_t)$$

But now this term becomes a problem.

But note that the last term is

$$\hat{\omega}_t = \frac{T-t}{T}\omega_0 + \frac{t}{T}\pi^*\omega_Y$$

$$\geq \frac{T-t}{T}\omega_0$$

Thus have then:

$$-\operatorname{tr}_\omega(\hat{\omega}_t) \leq -\frac{T-t}{T}\operatorname{tr}_\omega(\omega_0)$$

arithmetic-mean ineq.

$$\leq -n\frac{T-t}{T}\left(\frac{\omega_0^n}{\omega^n}\right)^{1/n}$$

$$\leq -nC'\frac{T-t}{T}\left(\frac{\Omega}{\omega^n}\right)^{1/n}$$

$$= -K\left(\frac{(T-t)^n\Omega}{\omega^n}\right)^{1/n}$$

So have then that

$$\left(\frac{\partial}{\partial t}-\Delta\right)(\log u - A\varphi) \leq A\log\frac{\Omega}{\omega^n} + An - AK\left(\frac{(T-t)^n\Omega}{\omega^n}\right)^{1/n}$$

$$= A\log\frac{(T-t)^n\Omega}{\omega^n} - AK\left(\frac{(T-t)^n\Omega}{\omega^n}\right)^{1/n}$$

$$- A\log(T-t) + An$$

Now since $\alpha \log \mu - \beta \mu^{1/n}$ is bounded above for ~~any~~ any α, β we get for some constant C'':

$$\left(\frac{\partial}{\partial t} - \Delta\right)(\log u - A\varphi) \leq C'' - A\log(T-t) + An$$

But now we're done because even though this blows up, we can find a <u>bounded</u> antiderivative!

$$\left(\frac{\partial}{\partial t} - \Delta\right)\underbrace{\left(\log u - A\varphi - A(T-t)\log(T-t) + A(T-t) - C''t\right)}\leq 0$$

Thus this whole thing is bounded above by it's values for $t = 0$. This gives a uniform bound for $\log u$ and thus a uniform bound for u!

Proof (ii): On $K \subset X \setminus E$, $\pi^*\omega_Y$ is a legit form and uniformly equivalent to ω_0 (dep on K) which is uniformly equivalent to Ω (i.e. $\omega_0^n \sim \Omega$). Thus by previous lemma:

- $\omega^n \leq C(K)\,\pi^*\omega_Y$

Using (i) of this lem, $\pi^*\omega_Y \leq C'\omega$. Thus the eigenvalues of ω are bounded above and below.

Standard local parabolic theory then apparently tells us that $\|\omega\|_{\mathcal{C}^i(K)} \leq C_{K\delta,i}$.

(iii) ~~Thus~~ Since φ converges pointwise to a bounded function in last lemma, ~~(...)~~ (ii) of this lemma says it converges smoothly and since the eigenvalues are bounded, ω_T is smooth Kähler of E.

(iv) I'm not sure what else to say here... $\boxtimes$

Estimates Using Exceptional Divisor Contraction

Up until this point, have not used that the flow contracts a divisor. Now we will use that to get better estimates near y_0.

- Near y_0 pick $\underset{\text{unit ball}}{D} \leftrightarrow \underset{\text{coord chart}}{U} \subset Y$, $0 \leftrightarrow y_0$
- $D = \{ z \in \mathbb{C}^n \mid \|z\|^2 < 1 \}$
- g_{Eucl} is standard metric (equiv to ω_Y on D!)
- $D_r \subset D$ is ball radius r wrt g_E

- $\tilde{D} = \{(z, \ell) \in D \times \mathbb{P}^{n-1} \mid z \in \ell$; i.e. $z^i \ell^j = z^j \ell^i\}$

- $\tilde{D}_i = (\ell_i \neq 0)$ has coord given by

$$\begin{cases} z_{(i)}{}^j = \dfrac{\ell^j}{\ell^i} & \text{if} \quad i \neq j \\ z_{(i)}{}^i = z^i & \text{if} \quad i = i \leftarrow \text{dumb} \end{cases}$$

- On $\tilde{D}_i$, E given by $(z_{(i)}{}^i \equiv 0)$
- $[E]$ has transition $\dfrac{z^i}{z^j}$ on $\tilde{D}_i \cap \tilde{D}_j$ and $S(z) = z^i$ on $\tilde{D}_i$ gives section, w/ $S(z) \equiv 1$ on $X \setminus \tilde{D}_{1/2}$
- h_1 on $[E]$ defined by

$$h_1 = \frac{\sum_{j=1}^{n} |\ell^j|^2}{|\ell^i|^2} \quad \text{on } \tilde{D}_i$$

$$\left.\begin{aligned} |S|^2_{h_1} &= |z_i|^2 \frac{|z^1|^2 + \cdots + |\hat{z}|^2}{|z^i|^2} \\ &= |z^1|^2 + \cdots + |z^n|^2 \end{aligned}\right\} \text{on } \tilde{D} \,!$$

- On $X \setminus \tilde{D}_{1/2}$ define h_2 by $|s|^2_{h_2} \equiv 1$
- On all X define h by $h = \rho_1 h_1 + \rho_2 h_2$

w/ POU so $h \equiv h_1$ on $\tilde{D}_{1/2}$ & $h \equiv h_2$ on $X \setminus \tilde{D}$

- On $\hat{D}_{1/2}$, $R(h) = R(h_1)$ is:

$$R(h) = -\frac{i}{2\pi} \partial\bar{\partial} \log\left(|z^1|^2 + \cdots + |z^n|^2\right)$$

<u>Lemma</u>: $\exists \varepsilon_0 > 0$ s.t. $\omega_X := \pi^* \omega_Y - \varepsilon_0 R(h)$ is Kähler.

Now compute: (on $\hat{D}_{1/2} \setminus E$)

- $\partial\bar{\partial}(\log(r^2)) = \frac{1}{r^2}\left[\delta_{jk}\, dz^j \wedge d\bar{z}^k - \frac{\bar{z}^j z^k}{r^2}\, dz^j \wedge d\bar{z}^k\right]$
- $\omega_X = \pi^* \omega_Y + \frac{i}{2\pi} \frac{\varepsilon_0}{r^2} \sum_{j,k} \left(\delta_{jk} - \frac{\bar{z}^j z^k}{r^2}\right) dz^j \wedge d\bar{z}^k$

<u>Lemma</u>: $\pi^* \omega_Y \leq \omega_X \leq C \dfrac{\pi^* \omega_Y}{|s|^2_h}$

<u>Pf</u>: This follows because <u>this</u> is positive definite.

For the ~~aa~~ second inequality, have that:

$$\omega_X = \pi^*\omega_Y + \frac{i}{2\pi}\frac{\varepsilon_0}{r^2}\sum_{j,k}\left(\delta_{jk} - \frac{\bar{z}^j z^k}{r^2}\right) dz^j \wedge d\bar{z}^k$$

$$\leq \pi^*\omega_Y + \frac{i}{2\pi}\frac{\varepsilon_0}{r^2}\sum dz^j \wedge d\bar{z}^j \qquad (\text{that positive!!})$$

$$= \frac{\pi^*\omega_Y \cdot r^2 + \frac{i}{2\pi}\varepsilon_0 \sum dz^j \wedge d\bar{z}^j}{r^2 \quad |s|_h^2}$$

But now note that $\pi^*\omega_Y \cdot r^2 \leq \pi^*\omega_Y \quad (r^2 \leq 1)$ and that in the z-coordinates both $\pi^*\omega_Y \Leftrightarrow \omega_Y$ and $\sum dz^j \wedge d\bar{z}^j$ are uniformly equivalent. So get:

$$\omega_X \leq C \frac{\pi^*\omega_Y}{|s|_h^2} \qquad \boxed{}$$

<u>Lemma:</u> $\frac{1}{C'}\pi^*\omega_Y \leq \omega_0 \leq C' \frac{\pi^*\omega_Y}{|s|_h^2}$

<u>Pf:</u> Follows because ω_0 unif equiv to ω_X (both fixed!).

So we have that $\omega_X \leq C \dfrac{\pi^*\omega_Y}{|s|_h^2}$. Now we'd like a similar bound for $\omega(t)$!

Lemma 2.5 : $\exists\, \delta > 0$, C uniform :

(i) $\omega(t) \leq \dfrac{C}{|s|_h^2} \pi^*\omega_Y$

(ii) $\omega(t) \leq \dfrac{C}{|s|_h^{2(1-\delta)}} \omega_0$

Pf : Fix $0 < \varepsilon \leq 1$ and apply max principal to :

$$Q_\varepsilon = \log tr_{\omega_0}(\omega) + A \log\left(|s|_h^{2+2\varepsilon} tr_{\pi^*\omega_Y}(\omega)\right) - A^2\varphi$$

If we can get a bound $Q_\varepsilon \leq C$ ind of ε, then :

$$\log\left(tr_{\omega_0}(\omega)\right) + A \log\left(|s|_h^2 \, tr_{\pi^*\omega_Y}(\omega)\right) \leq C,$$

since φ is already bounded.

Then have by previous lemma,

$$\omega_0 \leq C' \frac{\pi^* \omega_Y}{|s|_h^2}$$

$$\Rightarrow \operatorname{tr}_{\omega_0} \omega(t) \geq \frac{1}{C'} |s|_h^2 \operatorname{tr}_{\pi^* \omega_Y} \omega(t)$$

and therefore that

$$\tilde{A} \log \left(|s|_h^2 \operatorname{tr}_{\pi^* \omega_Y} (\omega) \right) \leq C'$$

which would give us the desired bound.

So back to Q_ε. Once again using

- $|s|_h^2 \operatorname{tr}_{\pi^* \omega_Y} (\omega) \leq C \operatorname{tr}_{\omega_0} (\omega)$

Get that:

$$Q_\varepsilon \leq A' \log \operatorname{tr}_{\omega_0} (\omega) + A' \log (|s|_h^2) \cdot \varepsilon - A^2 \varphi.$$

So if you fix t, then $Q_\varepsilon \to -\infty$ as $z \to E$.

Now suppose there exists $(x_0, t_0) \in X \setminus E \times (0,T)$ w/ $\sup_{X \setminus E \times [0,t_0]} = Q_\varepsilon(x_0,t_0)$. Then at this point,

switched!

Cao's computation

this I never checked!

$$
\begin{cases}
0 \le \left(\frac{\partial}{\partial t} - \Delta\right) Q_\varepsilon \\
\le C \operatorname{tr}_\omega(\omega_0) - A \operatorname{tr}_\omega\left(A\hat{\omega}_{t_0} - (1+\varepsilon)R(h) - C'\pi^*\omega_Y\right) \\
\quad - A^2 \log \frac{\omega^n}{\Omega} + A^2 n,
\end{cases}
$$

where C, C' are uniform. Now can choose A large ind of ε & t_0 so that

$$A\left(A\hat{\omega}_{t_0} - (1+\varepsilon)R(h) - C'\pi^*\omega_Y\right) \ge (C+1)\omega_0$$

Then at (x_0, t_0) have

$$\left(A^2 \log \frac{\omega^n}{\Omega} + \operatorname{tr}_\omega(\omega_0)\right)(x_0,t_0) \le C$$

But note that:

$$A^2 \log \omega^n + \frac{1}{2} \operatorname{tr}_\omega(\omega_0)$$

$$= A^2 \log \omega^n + \frac{1}{2} \frac{n\omega^{n-1} \wedge \omega_0}{\omega^n}$$

at (x_0, t_0) fixed no matter ~~what~~ what ω^n is

(Lemma 2)

$$\geq A^2 \log \omega^n + c^n \frac{n}{2} \frac{(\pi^* \omega_Y)^{n-1} \wedge \omega_0}{\omega^n}$$

$$\geq -C,$$

since $\log y + \frac{c}{y}$ bounded below. Thus have in fact that

$$\operatorname{tr}_\omega(\omega_0)(x_0, t_0) \leq C.$$

Then have at (x_0, t_0) that

$$\operatorname{tr}_{\omega_0}(\omega) = \frac{n\omega_0^{n-1} \wedge \omega}{\omega_0^n} = \left(\frac{\omega^n}{\omega_0^n}\right) \frac{n\omega_0^{n-1} \wedge \omega}{\omega^n}$$

$$\overset{??}{\leq} \left(\frac{\omega^n}{\omega_0}\right)\left(\frac{n\omega^{n-1} \wedge \omega_0}{\omega^n}\right)^{n-1} \frac{1}{(n-1)!}$$

$$\leq C.$$

So then we get that

$$\left(|s|_h^2 \, tr_{\pi^* \omega_Y}(\omega)\right)(x_0, t_0) \leq C$$

and that $Q_\varepsilon \leq C$. So this gives (i).

So now that we have that

$$\log tr_{\omega_0}(\omega) + A \log\left(|s|_h^2 \, \underset{\rotatebox{90}{$\leq$}}{tr_{\pi^* \omega_Y}(\omega)}\right) \leq C$$

$$\Rightarrow \log tr_{\omega_0}(\omega) + A \log\left(|s|_h^2 \, tr_{\omega_0}(\omega)\right) \leq C$$

$$\Rightarrow |s|_h^{2A} \, tr_{\omega_0}(\omega)^{A+1} \leq C'$$

$$\Rightarrow tr_{\omega_0}(\omega) \leq \frac{C''}{|s|_h^{\frac{2A}{A+1}}}$$

$\delta = \frac{1}{A+1}$

and since $\frac{2A}{A+1} = 2(1-\delta)$, we have (ii) as well.

Now look at $z^j \frac{\partial}{\partial z^j}$ on D. This can be pulled back to a holo vector field on $\tilde{D}$ vanishing on E to order 1.

<u>Lemma 2.6a</u>: $|V|^2_\omega \leq C|s|_h$

<u>Remark</u>: From last lemma, we know

- $$|V|^2_\omega \leq \frac{c}{r^2 r^{-2\delta}} |V|^2_{\omega_0}$$
$$\leq c' r^{2\delta},$$

w/ 2δ pretty small. This lemma allows us to bust it up to 1!

<u>Proof of Lemma</u>:

- $$\omega_X = \pi^* \omega_Y + \frac{i}{2\pi} \frac{\varepsilon_0}{r^2} \sum_{i,j=1}^{n} \left(\delta_{ij} - \frac{\bar{z}^i \bar{z}^j}{r^2} \right) dz^i \wedge d\bar{z}^j$$
- this evaluates to 0 on $z^i \frac{\partial}{\partial z^i}$

$$\Rightarrow \quad |V|^2_{\omega_X} = |V|^2_{\omega_Y} \sim r^2$$
$$|V|^2_{\omega_0}$$

Thus have:

$$\circ \quad \frac{1}{C} r^2 \le |V|^2_{\omega_0} \le C r^2$$

Thus for fixed t, have:

$$\circ \quad \frac{1}{C(t)} r^2 \le |V|^2_{\omega} \le C(t) r^2$$

Now define $\tilde{\omega}_Y = \omega_{Eucl}$ on D and extend to smooth metric on Y.

Define Q_ε by the following:

$$\circ \quad Q_\varepsilon = \log\left(|V|^{2+2\varepsilon}_{\omega} \operatorname{tr}_{\pi^* \tilde{\omega}_Y} \omega\right) - t$$

<u>Goal</u>: Show Q_ε uniformly bounded above ind. ε.

Then would have:

$$|V|^2_{\omega} \operatorname{tr}_{\pi^* \tilde{\omega}_Y} \omega \le C$$

Lemma 2.4 says $\frac{1}{C} \pi^* \omega_Y \le \omega_0$ and so

$$|V|^2_{\omega} \operatorname{tr}_{\omega_0} \omega \le C'$$

Now using the computation on (CS) have that

$|v|_\omega^2 \leq (\mathrm{tr}_{\omega_0}\omega)|v|_{\omega_0}^2$ and thus that

$$|v|_\omega^4 \leq C'|v|_{\omega_0}^2.$$

But since $|v|_{\omega_0}^2 \leq C r^2$ we get

$$|v|_\omega^2 \leq C r = C|s|_h.$$

Thus if we can show Q_ε is bounded above ind of ε we get our bound.

Our earlier stuff tells us that:

- $|s|_h^2 \,\mathrm{tr}_{\pi^*\omega_Y}(\omega) \leq C\,\mathrm{tr}_{\omega_0}(\omega)$
- $\frac{1}{C(t)}|s|_h^2 \leq |v|_\omega^2 \leq C(t)|s|_h^2$

Thus:

$$|v|_\omega^{2\varepsilon}|v|_\omega^2 \,\mathrm{tr}_{\pi^*\tilde{\omega}_Y}\omega \leq c'|s|_h^{2\varepsilon}|s|_h^2\,\mathrm{tr}_{\pi^*\omega_Y}(\omega)$$
$$\leq c''\,\mathrm{tr}_{\omega_0}(\omega(t))\,|s|_h^{2\varepsilon}$$

Thus if time is fixed, $Q_\varepsilon \to 0$ as $x \to E$.

Note that this is where the extra $|s|_h^{2\varepsilon}$ is needed.

On $X \setminus \tilde{D}_{1/2}$, ~~the~~ ω converges nicely so Q_ε is uniformly bounded above indpendent of ε in $X \setminus \tilde{D}_{1/2}$. Now need to show that Q_ε is uniformly bounded above inside $\tilde{D}_{1/2} \setminus \{0\}$.

So we need to rule out a maximum point for $t \neq 0$.

I.e. assume

$$\sup_{X \setminus E \times [0,t_0]} Q_\varepsilon = Q_\varepsilon(x_0, t_0)$$

with $(x_0, t_0) \in D_{1/2} \setminus \{0\} \times (0, T)$. Now apply a computation of Cao (used on page 22):

$$\left(\frac{\partial}{\partial t} - \Delta\right) \log \operatorname{tr}_{\tilde{\omega}} \omega = \frac{1}{\operatorname{tr}_{\tilde{\omega}} \omega} \left(g^{i\bar{j}} \tilde{R}_{\bar{j}i}{}^{k\bar{\ell}} g_{\bar{\ell}k} - g^{i\bar{j}} \tilde{g}^{k\bar{\ell}} g^{p\bar{q}} \tilde{\nabla}_i g_{\bar{q}k} \tilde{\nabla}_{\bar{j}} g_{\bar{\ell}p} + \frac{|\nabla \operatorname{tr}_{\tilde{\omega}} \omega|^2}{\operatorname{tr}_{\tilde{\omega}} \omega} \right)$$

But in our case, $\hat{\omega} = \pi^* \tilde{\omega}_Y$ which has vanishing curvature inside $D_{1/2}$! Thus we get

$$\left(\frac{\partial}{\partial t} - \Delta\right) \mathrm{tr}_{\pi^*\tilde{\omega}_Y}(\omega) \leq 0.$$

Next we'd like to compute : $\left(\frac{\partial}{\partial t} - \Delta\right)\left(\log |v|^2_\omega\right)$.

- $$\frac{\partial}{\partial t} \log\left(|v|^2_\omega\right) = \frac{\frac{\partial}{\partial t}\left(\omega_{\bar{k}j} v^j \bar{v}^k\right)}{|v|^2_\omega} \overset{(KRF)}{=} \frac{-R_{\bar{k}j} v^j \bar{v}^k}{|v|^2_\omega}$$

- $$\Delta\left(\log |v|^2_\omega\right) = \frac{1}{|v|^2_\omega}\left(\Delta\left(|v|^2_\omega\right) - \frac{|\nabla(|v|^2_\omega)|^2_\omega}{|v|^2_\omega}\right)$$

- $$\begin{aligned}\Delta\left(|v|^2_\omega\right) &= + \omega^{a\bar{b}} \partial_a \partial_{\bar{b}} \left(\omega_{\bar{k}j} v^j \bar{v}^k\right) \\ &= + \omega^{a\bar{b}} \partial_a \partial_{\bar{b}}(\omega_{\bar{k}j}) v^j \bar{v}^k + \omega^{a\bar{b}} \omega_{\bar{k}j} \partial_a v^j \partial_{\bar{b}} \bar{v}^k \\ &\quad + \omega^{a\bar{b}}\left(\cancel{\partial_a(\omega_{\bar{k}j})}^{\,0} v^j \partial_{\bar{b}} \bar{v}^k + \cancel{\partial_{\bar{b}}(\omega_{\bar{k}j})}^{\,0} \partial_a v^j \bar{v}^k\right)\end{aligned}$$

Now in normal coord at x_0, these two are zero and this is minus the Ricci form. Thus we have

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(\log |v|^2_\omega\right) = \frac{1}{|v|^2_\omega}\left(-\omega^{a\bar{b}} \omega_{\bar{k}j} \partial_a v^j \partial_{\bar{b}} \bar{v}^k + \frac{|\nabla(|v|^2_\omega)|^2_\omega}{|v|^2_\omega}\right)$$

Now you compute to get:

$$|\nabla |V|^2_\omega|^2_\omega = g^{j\bar{k}} \partial_j (g_{\bar{b}a} V^a \bar{V}^b) \partial_{\bar{k}} (g_{\bar{d}c} V^c \bar{V}^d)$$

$$\overset{\text{(Normal)}}{=} g^{j\bar{k}} g_{\bar{b}a} (\partial_j V^a) \bar{V}^b g_{\bar{d}c} V^c \partial_{\bar{k}} \bar{V}^d$$

$$\overset{\text{(Normal)}}{=} \sum_j \Big(\sum_k (\partial_j V^k) \overline{V^k} \Big) \Big(\sum_p \overline{(\partial_j V^p)} V^p \Big)$$

$$\leq \sum_j |\partial_j V|^2_\omega |V|^2_\omega = |V|^2_\omega g^{j\bar{k}} g_{\bar{b}a} (\partial_j V^a) \overline{(\partial_k V^b)}$$

Thus we actually have that

$$\left(\frac{\partial}{\partial t} - \Delta\right)(\log |V|^2_\omega) \leq 0.$$

Thus in conclusion, at (x_0, t_0), $0 \leq \left(\frac{\partial}{\partial t} - \Delta\right)(Q_\varepsilon) \leq -1$.

Thus Q_ε must be bounded above. ∎

<u>Lemma 2.66</u>: Define $W = \frac{1}{r}\left(x^i \frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial y^i}\right)$. Then

$$|W|^2_g \leq \frac{C}{r},$$

locally in $D_{1/2} \setminus 0$.

<u>Pf</u>: Use last lemma and $W = \mathrm{Re}\left(\frac{2}{r} V\right)$. ∎

Lemma 2.7 (i) For $0 < r < \frac{1}{2}$, $S_r \subset D$ the $2n-1$ sphere, $\mathrm{diam}_\omega(S_r) \le C$ ind of r.

(ii) For any $x \in D_{1/2} \setminus \{0\}$, $\mathrm{length}_\omega(\lambda(\lambda))$, the radial path, on $(0,1]$, is bdd above by $C|x|^{1/2}$.

Thus $\mathrm{Diam}_\omega(D_{1/2} \setminus \{0\})$ bounded above and $\mathrm{diam}_\omega(X) \le C$.

Proof: (i) Lemma 2.5 says $\omega \le \frac{C}{r^2} \pi^* \omega_Y$ and in the coord on $D \setminus \{0\}$, $\pi^*\omega_Y \sim \omega_{Eucl}$. Thus

$$\omega \le \frac{C}{r^2} \omega_{Eucl}.$$

Now w/ $\iota_r : S_r \hookrightarrow D$ the inclusion, have

$$d_{\iota_r^* g}(p,q) \le \frac{\sqrt{C}}{r} d_{\iota_r^* g_{Eucl}}(p,q)$$

$$\le \frac{\sqrt{C}}{r} \cdot \pi r = \sqrt{C}\pi.$$

(ii) Now the path $\gamma(\lambda) = \lambda x$ has derivative rW.

Thus

$$\int_0^1 \sqrt{g_{\gamma(\lambda)}(\gamma'(\lambda), \gamma'(\lambda))}\, d\lambda = \int_0^1 r \sqrt{|W|_g^2}\, d\lambda \le C r^{1/2}.$$

Now (i) and (ii) combined give diameter bound for $D_{1/2} \setminus \{0\}$. Since on $X \setminus \tilde{D}_{1/2}$ we have nice convergence, we have a diameter bound on $X \setminus E$.

Now only need to check for a diameter bound if at least one of p & q lie in E.

But then let $p_i \to p$ & $q_j \to q$ wrt to a fixed metric and then the uniform bound on $d_\omega(p_i, q_j)$ gives the uniform bound of $d_\omega(p,q)$.

Gromov - Hausdorff Convergence

We know that $\omega(t) \to \omega_T$ smoothly on $X \setminus E$. Thus we can view ω_T as a smooth Kähler metric on $Y \setminus \{y_0\}$. Let g_T be the corresponding metric on $Y \setminus \{y_0\}$. Now define g_T on Y by extending to whatever you want.

Def: Define a distance function $d_T : Y \times Y \to \mathbb{R}$ by

$$d_T(y_1, y_2) = \inf_{\gamma} \int_0^1 \sqrt{g_T(\gamma'(s), \gamma'(s))}\, ds,$$

where γ are piecewise smooth paths. By our diameter bounds on $\omega(t)$, we know this is finite.

Lemma 3.1: (Y, d_T) is a compact metric space.

Proof: • First wnt $y_1 \neq y_2 \Rightarrow d_T(y_1, y_2) > 0$.

Assuming $y_1 \neq y_0$, choose a small ball about y_1 not containing y_2. In this ball ω_T is Kähler so there is a minimal escape velocity. Thus done.

• Now need to show compactness. Let $\{y_j\}$ be a sequence in Y. Then, with respect to ω_Y, there is a convergent subsequence, $y_j \to y$.

(i) If $y \neq y_0$, then the y_j eventually lie outside a neighborhood of y_0, where ω_Y and ω_T are uniformly equiv. Thus done.

(ii) If $y = y_0$, then the y_j are eventually in D. Here can assume $y_j \to 0$ w/ ω_{Eucl}. But the radial paths have lengths bounded by $c|y|^{1/2}$ by lemma 2.7. Thus $d(y_0, y_j) \to 0$.

$\boxed{14}$

Thus $y_j \to y$ wrt d_T and <u>hence</u> compact!

<u>Lemma 3.2</u>: Let $d_\omega = d_{\omega(t)}$ be distance function with respect to ω. Then $\exists C$ unif s.t.:

$$d_\omega(p,q) \le C(T-t)^{1/3}, \quad \forall p,q \in E.$$

<u>Proof</u>: First assume $\dim E = 1$. Then have

$$\int_E \omega(t) = \frac{1}{T}\int_E \left((T-t)\omega_0 + \cancel{t\pi^*\omega_Y}^{\,\to 0}\right)$$

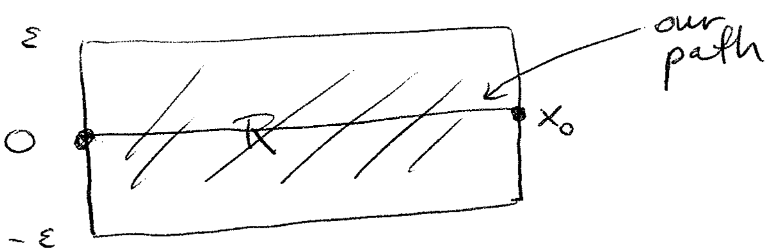
$$= \frac{T-t}{T}\int_E \omega_0$$

$$= C(T-t)$$

Without loss of generality, assume p, q lie inside a chart $\mathcal{U} \subset \mathbb{C}$ corresponding to a ~~unit~~ radius 2 ball and that $p \leftrightarrow (0,0)$, $q \leftrightarrow (r_0, 0)$, w/ $0 < r_0 < 1$.

For small ε, consider

$$\mathcal{R} = \{ (x,y) \mid 0 \le x \le x_0, \ -\varepsilon \le y \le \varepsilon \} \subset \mathbb{R}^2 = \mathbb{C}$$



In R, $g_0 \sim g_{Eucl}$ "$\hat{g}_0$" so:

det $\hat{g}_0$

$$\int_{-\varepsilon}^{\varepsilon} \left(\int_0^{x_0} (\mathrm{tr}_{\hat{g}_0} g)\, dx \right) dy = \int_R (\mathrm{tr}_{\hat{g}_0} g)\, dx\, dy \le C(T-t)$$

Thus there is a $y' \in (-\varepsilon, \varepsilon)$ such that

$$\int_0^{x_0} (\mathrm{tr}_{\hat{g}_0} g)(x, y')\, dx \le \frac{C}{\varepsilon}(T-t).$$

Picking ε intelligently, we consider $\varepsilon = (T-t)^{\alpha}$ for α positive. Then

$$\int_0^{x_0} (\mathrm{tr}_{\hat{g}_0}\, g)(x, y')\,dx \le C\,(T-t)^{1-\alpha}$$

So now look at the path $s \mapsto (s, y')$ from $p' \to q'$:

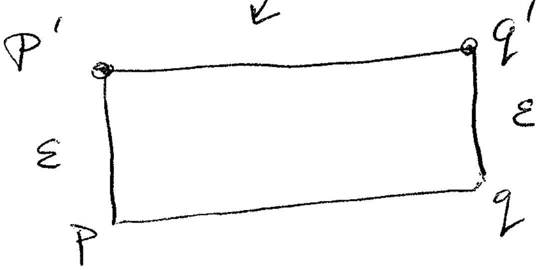
$$d_\varepsilon(p', q') \le \int_0^{x_0} \left(\sqrt{g(\partial_x, \partial_x)}\right)(x, y')\,dx$$

$$= \int_0^{x_0} \left(\sqrt{\mathrm{tr}_{\hat{g}_0}\, g}\,\sqrt{\hat{g}_0(\partial_x, \partial_x)}\right)(x, y')\,dx$$

$$\le \left(\int_0^{x_0} (\mathrm{tr}_{\hat{g}_0}\, g)(x, y')\,dx\right)^{1/2} \left(\int_0^{x_0} (\hat{g}_0(\partial_x, \partial_x))(x, y')\,dx\right)^{1/2}$$

$$\le C\,(T-t)^{\frac{1-\alpha}{2}}$$

Picture:



$\varepsilon \leftarrow (T-t)^{\alpha}$

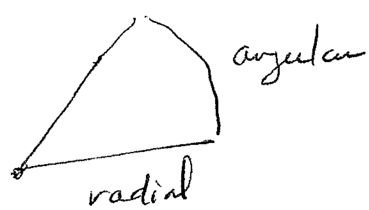
So need to estimate $d_\omega(P, P')$ & $d_\omega(q, q')$.

Now we will leave the divisor because we have metric information for $\omega(t)$ near E depending on the Euclidean distance between p & p' inside E.

In Lemma 2.7 we used a spot w/

$$d^*_{L_r g}(p,p') \le \frac{\sqrt{C}}{r} \underbrace{\pi r}_{\text{bad}} = \sqrt{C}\,\pi$$

But πr can be replaced by $d^*_{L_r g_{Eucl}}(p,p')$ which we know is on the order of ε. The radial path can be made as small as we want.



So we have heuristically a distance of

$$\text{“}(T-t)^{\frac{1-\alpha}{2}} + (T-t)^{\alpha} + (T-t)^{\alpha}\text{”}$$

But we want exponent as large as possible. Thus we need $\alpha = \frac{1}{3}$ and we're done. $\boxed{\text{Q}}$

Lemma: For any $\varepsilon > 0$ there exists $\delta_0 > 0$ and $T_0 \in [0, T)$ s.t.

$$\operatorname{diam}_{d_T} D_{\delta_0} < \varepsilon$$ ← so d_T gives finer topology

and

$$\operatorname{diam}_{\omega(t)} \pi^{-1}(D_{\delta_0}) < \varepsilon, \quad \text{for all } t \in [T_0, T).$$

Proof: Fix ε and then pick δ_0 small so that radial paths are less than $\frac{\varepsilon}{2}$ (i.e. $p \to 0$, $\xi \to 0$).

The second part comes from Lemmas 2.7 & 3.2.

Remark: This means the topologies are the same. Lemma 2.2. (i) gives the other direction. Thus homeomorphism!

Now the goal is to get the Gromov-Hausdorff convergence.

Def: $d_{GH}(X,Y)$ is infimum of ε s.t.:

- $\exists F : X \to Y$, $G : Y \to X$ (not nec continuous)
- ε-isometry → $|d_X(x_1, x_2) - d_Y(F(x_1), F(x_2))| \le \varepsilon, \quad \forall x_1, x_2 \in X$
- ε-psuedoinverse → $d_X(x, G(F(x))) \le \varepsilon, \quad \forall x$
- plus symmetric

For us $F = \pi$ and $G = \pi^{-1}$ w/ $\pi^{-1}(y_0) \in E$ chosen arbitrarilly but fixed.

Now want to show that we can choose t so that X & Y are w/in ε distance of each other.

First: • $d_T(y, F(G(y))) = 0$, $\forall y$

• $d_{\omega(t)}(x, G(F(x))) = 0$, $\forall x \in X \setminus E$

So choose t close to T so $\operatorname{diam}_{\omega(t)}(E) < \varepsilon$!

Second: Want: $\left| d_{\omega(t)}(x_1, x_2) - d_T(F(x_1, x_2)) \right| \leq \varepsilon$

Using Lemma 3.3, pick $\delta_0 > 0$ and $T_0 \in [0, T)$ so that $\forall x_1, x_2 \in \pi^{-1}(\overline{D_{\delta_0}})$, $\forall t \in [T_0, T)$ we have

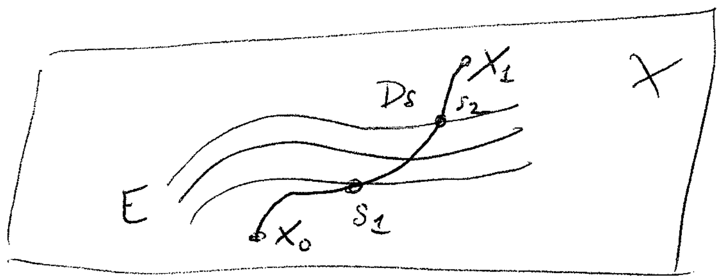
$$d_{\omega(t)}(x_1, x_2) + d_T(F(x_1), F(x_2)) \in \frac{\varepsilon}{2}.$$

Thus can assume x_1 & x_2 lie outside $\pi^{-1}(D_{\delta_0})$.

So there exists a smooth path $\gamma : [0,1] \to$, $F(x_1) \to F(x_2)$

$$\left| d_T(F(x_1), F(x_2)) - \int_0^1 \sqrt{\tilde{g}_T(\gamma'(s), \gamma'(s))}\, ds \right| < \frac{\varepsilon}{4}.$$

Then have picture:



Actually picture should be in Y!

Pick s_1 entry, s_2 exit:

$$d_{\omega(t)}(x_1,x_2) \leq \int_0^{s_1} \sqrt{g(t)(\gamma'(s),\gamma'(s))}\,ds + \int_{s_2}^{1} \sqrt{g(t)(\gamma'(s),\gamma'(s))}\,ds$$

$$+ \underbrace{d_{\omega(t)}\left(F^{-1}(\gamma(s_1)), F^{-1}\gamma(s_2)\right)}_{< \frac{\varepsilon}{2} \text{ by last page}}$$

These guys are unif controlled! Thus get

$$d_{\omega(t)}(x_1,x_2) - d_Y(F(x_1),F(x_2)) \leq \varepsilon.$$

Lower & symmetric arguments are similar.

Higher Order Estimates

Define endomorphism $H = H(t)$ by $H^i_\ell = g_0^{i\bar{j}} g_{\bar{j}\ell}$

and define $S = S(t)$ by

$$S = |\nabla_g H H^{-1}|^2_g$$

Prop 4.1: $\exists \alpha, C$ positive so that for $t \in [0,T)$

$$S \leq \frac{C}{|s|_h^{2\alpha}}$$

Prop 4.2: For each integer $m \geq 0$, $\exists C_m, \alpha_m > 0$ such that for $t \in [0,T)$,

$$|\nabla_{\mathbb{R}}^m Rm(g)| \leq \frac{C_m}{|s|_h^{2\alpha_m}},$$

when $\nabla_{\mathbb{R}} = \frac{1}{2}(\nabla + \overline{\nabla})$ wrt g.

Corollary 4.1: For each $m \geq 0$, $\exists C_m, \alpha_m > 0$ s.t. for $t \in [0,T)$,

$$|(\nabla^0_{\mathbb{R}})^m g(t)|_{g_0} \leq \frac{C_m}{|s|_h^{2\alpha_m}},$$

when ∇^0 is respect to fixed g_0.

Continuing the Kähler Ricci Flow

We have that

$$\omega_T = \hat{\omega}_T + \frac{i}{2\pi}\partial\bar\partial \varphi_T \geq 0, \quad \text{on } X,$$

where $\hat{\omega}_T = \pi^* \omega_Y$, $\varphi(t) \to \varphi_T$ pointwise, smoothly off of E.

Lemma 5.1: $\varphi_T|_E \equiv$ constant

Proof: $\omega_T|_E = \frac{i}{2\pi}\partial\bar\partial\varphi|_E \geq 0$. Thus φ_T is constant on E.

Remark: Thus we have $\varphi_T = \pi^* \psi_T$, where ψ_T is smooth off $\{y_0\}$ and is bounded.

Define: $\omega' := \omega_Y + \frac{i}{2\pi}\partial\bar\partial \psi_T \geq 0$. Thus ω' is the pushdown current to Y.

Lemma: There exists $p > 1$ such that $\frac{(\omega')^n}{(\omega_Y)^n} \in L^p(Y)$.

Proof: The boundedness of ψ_T and smoothness away from y_0 means $(\omega')^n/(\omega_Y)^n$ has no mass at y_0. Thus it is in $L^1(Y)$.

So then:

$$\int_Y \left(\frac{(\omega')^n}{\omega_Y^n}\right)^p \omega_Y^n = \int_{Y \setminus \{y_0\}} \text{———}$$

$$= \int_{X \setminus E} \left(\frac{\omega_T^n}{(\pi^*\omega_Y)^n}\right)^p (\pi^*\omega_Y)^n$$

$$\leq C \int_{X \setminus E} \left(\frac{\Omega}{(\pi^*\omega_Y)^n}\right)^{p-1} \Omega$$

how does it work exactly? $\longrightarrow$ $(*)$

$$\leq C,$$

if p sufficiently small.

Cor: Kolodziej says φ_T must be continuous!

Pick $\chi \in c_1(K_X)$ smooth & closed. Then $\exists T' > T$ such that

$$\hat{\omega}_{t,Y} := \omega_Y + (t - T)\chi$$

remains Kähler. Now fix Ω_Y by

$$\frac{i}{2\pi}\partial\bar{\partial}\log\Omega_Y = \frac{\partial\hat{\omega}_t}{\partial t} = \chi \in c_1 (K_Y).$$

For fixed K large, ε small define Ω_ε by

$$\Omega_\varepsilon = \left(\pi\big|_{X\setminus E}^{-1}\right)^* \left(\frac{|s|_h^{2K}\, \overset{\omega'}{\omega^n}\, \overset{1?}{(T-\varepsilon)}}{\varepsilon + |s|_h^{2K}} \right) + \varepsilon\, \Omega_Y$$

on $Y \setminus \{y_0\}$, and $\Omega_\varepsilon\big|_{y_0} = \varepsilon\, \Omega_Y\big|_{y_0}$.

Now define $\psi_{T,\varepsilon}$ by:

$$\begin{cases} \bullet\ \left(\omega_Y + \frac{i}{2\pi}\partial\bar\partial \psi_{T,\varepsilon}\right)^n = C_\varepsilon\, \Omega_\varepsilon \\ \bullet\ \sup_Y \left(\psi_{T,\varepsilon} - \psi_T\right) = \sup_Y \left(\psi_T - \psi_{T,\varepsilon}\right) \\ \bullet\ \cancel{\varepsilon}\, C_\varepsilon \int_Y \Omega_\varepsilon = \int_Y \omega_Y^n \end{cases}$$

Solutions lie in $\cancel{C^k(X)}$ $C^k(Y) \cap C^\infty(Y \setminus \{y_0\})$.

Kolodziej's stability theorem tells us

$$\| \psi_{T,\varepsilon} - \psi_T \|_{L^\infty(Y)} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Define φ_ε by:

$$\begin{cases} \dot\varphi_\varepsilon = \log \dfrac{\left(\hat\omega_t + \frac{i}{2\pi}\partial\bar\partial \varphi_\varepsilon\right)^n}{\Omega_Y} \\ \varphi_\varepsilon\big|_T = \psi_{T,\varepsilon} \end{cases}$$

Proposition 5.1: $\exists \varphi$ in $\mathcal{C}^0([T,T']\times Y)\cap \mathcal{C}^\infty((T,T']\times Y)$,

- $\varphi_\varepsilon \to \varphi$ in $L^\infty([T,T']\times Y)$
- $\varphi_\varepsilon \to \varphi$ is $\mathcal{C}^\infty$ on compact subsets of $(T,T']\times Y$.
- φ is unique solution of

$$\varphi\big|_{t=T} = \phi_T$$

$$\dot{\varphi} = \log \frac{(\hat{\omega}_t + \frac{i}{2\pi}\partial\bar{\partial}\varphi)^n}{\Omega_Y}, \quad t\in (T,T']$$

Proof: Exercise left to the reader.

Lemma 5.3: $\exists\, C, \alpha > 0$ ind of ε s.t. on $Y\setminus\{y_0\}$,

$$\frac{|s|_h^{2\alpha}}{C}\omega_Y \le \omega_{T,\varepsilon} \le \frac{C}{|s|_h^{2\alpha}}\omega_Y$$

Fix N large. For each $0\le m\le N$ there exists $C_m, \alpha_m > 0$ s.t.

$$|(\nabla_{\mathbb{R}}^Y)^m g_{T,\varepsilon}|_{g_Y} \le \frac{C}{|s|_h^{2\alpha_m}},$$

where $\nabla_{\mathbb{R}}$ is real cov. deriv. wrt fixed g_Y.

Want :

$$g^{k\bar{\ell}}\partial_k\partial_{\bar{\ell}}\left(g^{i\bar{j}}\pi^{\alpha}_{i}\pi^{\bar{\beta}}_{\bar{j}}h_{\bar{\beta}\alpha}\right) = g^{i\bar{\ell}}g^{k\bar{j}}R_{\bar{\ell}k}\pi^{\alpha}_{i}\pi^{\bar{\beta}}_{\bar{j}}h_{\bar{\beta}\alpha}$$

$$+ g^{i\bar{j}}g^{k\bar{\ell}}\pi^{\alpha}_{i,k}\pi^{\bar{\beta}}_{\bar{j},\bar{\ell}}h_{\bar{\beta}\alpha} - g^{i\bar{j}}g^{k\bar{\ell}}S_{\bar{\delta}\gamma\bar{\beta}\alpha}\pi^{\alpha}_{i}\pi^{\bar{\beta}}_{\bar{j}}\pi^{\gamma}_{k}\pi^{\bar{\delta}}_{\bar{\ell}}$$

First:

- $$R_{\bar{\ell}k} = R_{\bar{\ell}k}{}^{c}{}_{c}$$
$$= -\partial_{\bar{\ell}}\left(g^{c\bar{d}}\partial_k(g_{\bar{d}c})\right)$$
$$= -g^{c\bar{d}}\partial_{\bar{\ell}}\partial_k(g_{\bar{d}c}) + g^{c\bar{a}}\partial_{\bar{\ell}}(g_{\bar{a}b})g^{b\bar{d}}\partial_k(g_{\bar{d}c})$$

- $$S_{\bar{\delta}\gamma\bar{\beta}\alpha} = -\partial_{\bar{\delta}}\left(h^{\eta\bar{\mu}}\partial_\gamma(h_{\bar{\mu}\alpha})\right)h_{\bar{\beta}\eta}$$
$$= -\partial_{\bar{\delta}}\partial_\gamma(h_{\bar{\beta}\alpha}) + h^{\eta\bar{\varphi}}\partial_{\bar{\delta}}(h_{\bar{\varphi}\phi})h^{\phi\bar{\mu}}\partial_\gamma(h_{\bar{\mu}\alpha})h_{\bar{\beta}\eta}$$
$$= -\partial_{\bar{\delta}}\partial_\gamma(h_{\bar{\beta}\alpha}) + \partial_{\bar{\delta}}(h_{\bar{\beta}\eta})h^{\eta\bar{\mu}}\partial_\gamma(h_{\bar{\mu}\alpha})$$

- $$\partial_{\bar{\ell}}\left(g^{i\bar{j}}\pi^{\alpha}_{i}\pi^{\bar{\beta}}_{\bar{j}}h_{\bar{\beta}\alpha}\right) = \partial_{\bar{\ell}}(g^{i\bar{j}})\pi^{\alpha}_{i}\pi^{\bar{\beta}}_{\bar{j}}h_{\bar{\beta}\alpha} + g^{i\bar{j}}\pi^{\alpha}_{i}\pi^{\bar{\beta}}_{\bar{j},\bar{\ell}}h_{\bar{\beta}\alpha}$$
$$+ g^{i\bar{j}}\pi^{\alpha}_{i}\pi^{\bar{\beta}}_{\bar{j}}\pi^{\bar{\delta}}_{\bar{\ell}}\partial_{\bar{\delta}}(h_{\bar{\beta}\alpha})$$
$$= -g^{i\bar{a}}\partial_{\bar{\ell}}(g_{\bar{a}b})g^{b\bar{j}}\pi^{\alpha}_{i}\pi^{\bar{\beta}}_{\bar{j}}h_{\bar{\beta}\alpha} + g^{i\bar{j}}\pi^{\alpha}_{i}\pi^{\bar{\beta}}_{\bar{j},\bar{\ell}}h_{\bar{\beta}\alpha}$$
$$+ g^{i\bar{j}}\pi^{\alpha}_{i}\pi^{\bar{\beta}}_{\bar{j}}\pi^{\bar{\delta}}_{\bar{\ell}}\partial_{\bar{\delta}}(h_{\bar{\beta}\alpha})$$

Now apply ∂_k to all that shit:

$$\Rightarrow \overbrace{-\partial_k\left(g^{i\bar a}\,\partial_{\bar\ell}(g_{\bar a b})\,g^{b\bar j}\,\pi^{\alpha}_{;i}\,\pi^{\bar\beta}_{\bar j}\,h_{\bar\beta\alpha}\right)}^{A} + \overbrace{\partial_k\left(g^{i\bar j}\,\pi^{\alpha}_{;i}\,\pi^{\bar\beta}_{\bar j,\bar\ell}\,h_{\bar\beta\alpha}\right)}^{B}$$

$$+ \underbrace{\partial_k\left(g^{i\bar j}\,\pi^{\alpha}_{;i}\,\pi^{\bar\beta}_{\bar j}\,\pi^{\bar\delta}_{\bar\ell}\,\partial_{\bar\delta}(h_{\bar\beta\alpha})\right)}_{C}\Big)$$

$$A = -\partial_k(g^{i\bar a})\,\partial_{\bar\ell}(g_{\bar a b})\,g^{b\bar j}\,\pi^{\alpha}_{;i}\,\pi^{\bar\beta}_{\bar j}\,h_{\bar\beta\alpha} - g^{i\bar a}\,\partial_k\partial_{\bar\ell}(g_{\bar a b})\,g^{b\bar j}\,\pi^{\alpha}_{;i}\,\pi^{\bar\beta}_{\bar j}\,h_{\bar\beta\alpha}$$

$$- g^{i\bar a}\,\partial_{\bar\ell}(g_{\bar a b})\,\partial_k(g^{b\bar j})\,\pi^{\alpha}_{;i}\,\pi^{\bar\beta}_{\bar j}\,h_{\bar\beta\alpha} - g^{i\bar a}\,\partial_{\bar\ell}(g_{\bar a b})\,g^{b\bar j}\,\pi^{\alpha}_{i,k}\,\pi^{\bar\beta}_{\bar j}\,h_{\bar\beta\alpha}$$

$$- g^{i\bar a}\,\partial_{\bar\ell}(g_{\bar a b})\,g^{b\bar j}\,\pi^{\alpha}_{;i}\,\pi^{\bar\beta}_{\bar j}\,\pi^{\gamma}_{k}\,\partial_\gamma(h_{\bar\beta\alpha})$$

$$= \underbrace{g^{i\bar c}\,\partial_k(g_{\bar c d})\,g^{d\bar a}\,\partial_{\bar\ell}(g_{\bar a b})\,g^{b\bar j}\,\pi^{\alpha}_{;i}\,\pi^{\bar\beta}_{\bar j}\,h_{\bar\beta\alpha}}_{=0 \text{ due to Kähler?}} - \underbrace{g^{i\bar a}\,\partial_k\partial_{\bar\ell}(g_{\bar a b})\,g^{b\bar j}\,\pi^{\alpha}_{;i}\,\pi^{\bar\beta}_{\bar j}\,h_{\bar\beta\alpha}}_{4}$$

$$+ \underbrace{g^{i\bar a}\,\partial_{\bar\ell}(g_{\bar a b})\,g^{b\bar c}\,\partial_k(g_{\bar c d})\,g^{d\bar j}\,\pi^{\alpha}_{;i}\,\pi^{\bar\beta}_{\bar j}\,h_{\bar\beta\alpha}}_{3} - \underbrace{g^{i\bar a}\,\partial_{\bar\ell}(g_{\bar a b})\,g^{b\bar j}\,\pi^{\alpha}_{i,k}\,\pi^{\bar\beta}_{\bar j}\,h_{\bar\beta\alpha}}_{=0 \text{ due to Kähler?}}$$

$$- \underbrace{g^{i\bar a}\,\partial_{\bar\ell}(g_{\bar a b})\,g^{b\bar j}\,\pi^{\alpha}_{;i}\,\pi^{\bar\beta}_{\bar j}\,\pi^{\gamma}_{k}\,\partial_\gamma(h_{\bar\beta\alpha})}_{=0 \text{ due to Kähler?}}$$

$$B = \partial_k(g^{i\bar j})\,\pi^{\alpha}_{;i}\,\pi^{\bar\beta}_{\bar j,\bar\ell}\,h_{\bar\beta\alpha} + \underbrace{g^{i\bar j}\,\pi^{\alpha}_{i,k}\,\pi^{\bar\beta}_{\bar j,\bar\ell}\,h_{\bar\beta\alpha}}_{2} + g^{i\bar j}\,\pi^{\alpha}_{;i}\,\pi^{\bar\beta}_{\bar j,\bar\ell}\,\pi^{\gamma}_{k}\,\partial_\gamma(h_{\bar\beta\alpha})$$

$$= \underbrace{-g^{i\bar c}\,\partial_k(g_{\bar c d})\,g^{d\bar j}\,\pi^{\alpha}_{;i}\,\pi^{\bar\beta}_{\bar j,\bar\ell}\,h_{\bar\beta\alpha}}_{=0 \text{ due to Kähler}} + \underbrace{g^{i\bar j}\,\pi^{\alpha}_{i,k}\,\pi^{\bar\beta}_{\bar j,\bar\ell}\,h_{\bar\beta\alpha}}_{2} + \underbrace{g^{i\bar j}\,\pi^{\alpha}_{;i}\,\pi^{\bar\beta}_{\bar j,\bar\ell}\,\pi^{\gamma}_{k}\,\partial_\gamma(h_{\bar\beta\alpha})}_{=0 \text{ due to Kähler?}}$$

$$C = \partial_k(g^{i\bar j})\,\pi^{\alpha}_{;i}\,\pi^{\bar\beta}_{\bar j}\,\pi^{\bar\delta}_{\bar\ell}\,\partial_{\bar\delta}(h_{\bar\beta\alpha}) + g^{i\bar j}\,\pi^{\alpha}_{i,k}\,\pi^{\bar\beta}_{\bar j}\,\pi^{\bar\delta}_{\bar\ell}\,\partial_{\bar\delta}(h_{\bar\beta\alpha})$$

$$+ g^{i\bar j}\,\pi^{\alpha}_{;i}\,\pi^{\bar\beta}_{\bar j}\,\pi^{\bar\delta}_{\bar\ell}\,\pi^{\gamma}_{k}\,\partial_\gamma\partial_{\bar\delta}(h_{\bar\beta\alpha}) \qquad \text{C2}$$

$$= \underbrace{-g^{i\bar c}\,\partial_k(g_{\bar c d})\,g^{d\bar j}\,\pi^{\alpha}_{;i}\,\pi^{\bar\beta}_{\bar j}\,\pi^{\bar\delta}_{\bar\ell}\,\partial_{\bar\delta}(h_{\bar\beta\alpha})}_{=0 \text{ due to Kähler?}} + \underbrace{g^{i\bar j}\,\pi^{\alpha}_{i,k}\,\pi^{\bar\beta}_{\bar j}\,\pi^{\bar\delta}_{\bar\ell}\,\partial_{\bar\delta}(h_{\bar\beta\alpha})}_{=0 \text{ due to Kähler?}} + \underbrace{g^{i\bar j}\,\pi^{\alpha}_{;i}\,\pi^{\bar\beta}_{\bar j}\,\pi^{\bar\delta}_{\bar\ell}\,\pi^{\gamma}_{k}\,\partial_\gamma\partial_{\bar\delta}(h_{\bar\beta\alpha})}_{1}$$

Expanding out the side we want we get:

$$\boxed{RHS} = \underbrace{-g^{i\bar{\ell}}g^{k\bar{j}}g^{c\bar{d}}\partial_{\bar{\ell}}\partial_k(g_{\bar{d}c})\pi^\alpha_i\pi^{\bar{\beta}}_{\bar{j}}h_{\bar{\beta}\alpha}}_{4} + \underbrace{g^{i\bar{\ell}}g^{k\bar{j}}g^{c\bar{a}}\partial_{\bar{\ell}}(g_{\bar{a}b})g^{b\bar{d}}\partial_k(g_{\bar{d}c})\pi^\alpha_i\pi^{\bar{\beta}}_{\bar{j}}h_{\bar{\beta}\alpha}}_{3}$$

$$+ \underbrace{g^{i\bar{j}}g^{k\bar{\ell}}\pi^\alpha_{i,k}\pi^{\bar{\beta}}_{\bar{j},\bar{\ell}}h_{\bar{\beta}\alpha}}_{2}$$

$$+ \underbrace{g^{i\bar{j}}g^{k\bar{\ell}}\partial_{\bar{\delta}}\partial_\gamma(h_{\bar{\beta}\alpha})\pi^\alpha_i\pi^{\bar{\beta}}_{\bar{j}}\pi^\gamma_k\pi^{\bar{\delta}}_{\bar{\ell}}}_{1} \underbrace{-g^{i\bar{j}}g^{k\bar{\ell}}\partial_{\bar{\delta}}(h_{\bar{\beta}\eta})h^{\eta\bar{\mu}}\partial_\gamma(h_{\bar{\mu}\alpha})\pi^\alpha_i\pi^{\bar{\beta}}_{\bar{j}}\pi^\gamma_k\pi^{\bar{\delta}}_{\bar{\ell}}}_{=0 \text{ due to Kähler?}}$$

Red = Terms which cancel on LHS & RHS

Blue = Conjecturall = 0 due to Kähler Condition

Yes! Kähler condition means that at each point there are holomorphic normal coordinates meaning $h_{\bar{\beta}\alpha}$, $g_{\bar{k}j}$ vanish to order 1. Since all those are pointwise statements we happy :).

Now want inequality: $\dfrac{|\nabla(u)|^2}{u} \leq g^{i\bar{j}}g^{k\bar{\ell}}\pi^\alpha_{i,k}\pi^{\bar{\beta}}_{\bar{j},\bar{\ell}}h_{\bar{\beta}\alpha}$

- $\partial_{\bar{\ell}}(u) = \partial_{\bar{\ell}}(g^{j\bar{k}}\pi^\alpha_j\pi^{\bar{\beta}}_{\bar{k}}h_{\bar{\beta}\alpha})$

$$= \underbrace{-g^{j\bar{b}}\partial_{\bar{\ell}}(g_{\bar{b}a})g^{a\bar{k}}\pi^\alpha_j\pi^{\bar{\beta}}_{\bar{k}}h_{\bar{\beta}\alpha})}_{=0 \text{ if contracted}} + g^{j\bar{k}}\pi^\alpha_j\pi^{\bar{\beta}}_{\bar{k},\bar{\ell}}h_{\bar{\beta}\alpha} + \underbrace{g^{j\bar{k}}\pi^\alpha_j\pi^{\bar{\beta}}_{\bar{k}}\pi^{\bar{\delta}}_{\bar{\ell}}\partial_{\bar{\delta}}(h_{\bar{\beta}\alpha})}_{=0 \text{ if contracted}}$$

$$\Rightarrow |\nabla u|^2 = g^{m\bar{\ell}}g^{j\bar{k}}\pi^\alpha_j\pi^{\bar{\beta}}_{\bar{k},\bar{\ell}}h_{\bar{\beta}\alpha}g^{a\bar{b}}\pi^\gamma_{a,m}\pi^{\bar{\delta}}_{\bar{b}}h_{\bar{\delta}\gamma}$$

So then need to have:

$$|\nabla(u)|^2 \le \left(g^{i\bar{j}} g^{k\bar{l}} \pi^{\alpha}_{i,k} \pi^{\bar{\beta}}_{\bar{j},\bar{l}} h_{\bar{\beta}\alpha}\right) u$$

$$\Longleftrightarrow g^{m\bar{l}} g^{j\bar{k}} \pi^{\alpha}_{j} \pi^{\bar{\beta}}_{\bar{k},\bar{l}} h_{\bar{\beta}\alpha} g^{a\bar{b}} \pi^{\gamma}_{a,m} \pi^{\bar{\delta}}_{b} h_{\bar{\delta}\gamma}$$

$$\le g^{i\bar{j}} g^{k\bar{l}} \pi^{\alpha}_{i,k} \pi^{\bar{\beta}}_{\bar{j},\bar{l}} h_{\bar{\beta}\alpha} g^{a\bar{b}} \pi^{\gamma}_{a} \pi^{\bar{\delta}}_{\bar{b}} h_{\bar{\delta}\gamma}$$

This must follow by fancy Cauchy-Schwarz! (??? Still need to check this! ???)

Pointless computation:

$$(|V|^2_\omega)^2 = (\omega_{\bar k j} V^j \bar V^k)^2$$

$$= \left(\omega_0^{\ell\bar b}(\omega_{0\bar k \ell}\bar V^k)(\omega_{\bar b j}V^j)\right)^2$$

$$= \left(\omega_0^{\ell \bar b}(\omega_{0\bar k\ell}\bar V^k)\overline{(\omega_{\bar j b}\bar V^j)}\right)^2$$

$$\overset{C\text{-}S}{\leq} \omega_0^{\ell\bar b}(\omega_{0\bar k\ell}\bar V^k)\overline{(\omega_{0\bar c b}\bar V^c)}\,\omega_0^{\alpha\bar\beta}(\omega_{\bar\gamma\alpha}\bar V^\gamma)\overline{(\omega_{\bar\delta\beta}\bar V^\delta)}$$

$$= \bar V^b \omega_{0\bar b c}V^c\cdot(\omega_0^{\alpha\bar\beta}\omega_{\bar\gamma\alpha})\omega_{\bar\beta\delta}V^\delta\bar V^\gamma$$

$$= |V|^2_{\omega_0}(\omega_0^{\alpha\bar\beta}\omega_{\bar\gamma\alpha})\omega_{\bar\beta\delta}V^\delta\bar V^\gamma$$

$$\leq |V|^2_{\omega_0}(\omega_0^{\alpha\bar\beta}\omega_{\bar\beta\alpha})\omega_{\bar\gamma\delta}V^\delta\bar V^\gamma$$

$$= |V|^2_{\omega_0}(\mathrm{tr}_{\omega_0}\omega)\,|V|^2_\omega.$$

The last step follows because all the extra terms you get are positive. Thus

$$|V|^2_\omega \leq (\mathrm{tr}_{\omega_0}\omega)\,|V|^2_{\omega_0}$$